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Warped product pointwise hemislant submanifolds whose ambient spaces are nearly para-Kaehler manifolds

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Abstract: In this article, we explain pointwise slant, pointwise hemislant, and warped product pointwise hemislant submanifolds whose ambient spaces are nearly para-Kaehler manifolds. Also, we obtain several theorems and examples. Subsequently, we get some results concerning the inequality.

Key words: Nearly para-Kaehler manifold, pointwise hemislant submanifold, warped product submanifold

1. Introduction

Slant submanifolds were explained by Chen in 1990 as a natural generalization of both invariant and antiinvariant submanifolds in Hermitian manifold [4]. Later, pointwise slant and pointwise hemislant submanifolds of different construction and (semi)-Riemannian manifolds are studied in [1, 6, 9].

Warped product manifolds were introduced by Bishop and O'Neill [3]. Warped product manifold $\mathcal{N}_x = \mathcal{N}_a \times_k \mathcal{N}_b$ is a product manifold $\mathcal{N}_a \times \mathcal{N}_b$ equipped by a Riemannian metric $\breve{g}_x = \breve{g}_1 + k^2 \breve{g}_2$ and function k is a warping function [3, 12]. The notion of warped product manifolds is generally used in differential geometry, theory of general relativity, string theory, and black holes [12].

Recently, Sahin worked warped product hemislant submanifolds whose ambient spaces are Kaehler manifolds [9]. He demonstrated that warped product pseudoslant $N_b^{\perp} \times_k N_a^{\theta}$ submanifolds do not exist and he obtained a characterization and an inequality for the existing of warped product of the form $N_a^{\theta} \times_k N_b^{\perp}$ of Kaehler manifold. Later, Gündüzalp studied warped product pointwise hemislant submanifolds whose ambient spaces are para-Kaehler manifolds [6].

Then, Tachibana studied manifolds initially [10]. These manifolds are nearly Kaehler manifolds. For example, S^6 (six dimensional sphere) is a example of nearly Kaehler non-Kaehler manifold.

Every nearly para-Kaehler manifold is not a para-Kaehler, but every para-Kaehler manifold is a nearly para-Kaehler. This generalization is correct, so we provide examples of both nearly para-Kaehler and para-Kaehler manifolds, and we investigate warped product pointwise hemislant submanifolds whose ambient spaces are nearly para-Kaehler manifolds in this article.

This article is organized as follows. In Section 2, we describe pointwise slant submanifolds and present some examples and some basic results of these manifolds. In Section 3, we introduce proper pointwise hemislant

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submanifolds whose ambient spaces are nearly para-Kaehler manifolds and discuss their properties. In Section 4, we describe warped product pointwise hemislant submanifolds whose ambient spaces are nearly para-Kaehler manifolds. Additionally, we obtain some basic results and examples. In Section 5, we also obtain an inequality.

2. Preliminaries

Let $\bar{\mathcal{N}}_x$ be a $(2\bar{n})$-dimensional almost paracomplex metric structure. If it is provided with the structure $(\mathcal{P}, \breve{g}_1)$, where $\mathcal{P}$ is a tensor field of type $(1, 1)$ and $\breve{g}_1$ is expressed as a semi-Riemannian metric

$$\mathcal{P}^2 = \mathcal{I}, \quad \breve{g}_1(\mathcal{P}\mathcal{X}_a, \mathcal{P}\mathcal{Y}_b) = -\breve{g}_1(\mathcal{X}_a, \mathcal{Y}_b) \tag{1}$$

and

$$\breve{g}_1(\mathcal{P}\mathcal{X}_a, \mathcal{Y}_b) = -\breve{g}_1(\mathcal{X}_a, \mathcal{P}\mathcal{Y}_b) \tag{2}$$

for all vector fields $\mathcal{X}_a, \mathcal{Y}_b$ on $\bar{\mathcal{N}}_x$.

Let $\mathcal{T}\bar{\mathcal{N}}_x$ state the tangent bundle of $\bar{\mathcal{N}}_x$ and $\bar{\nabla}$, the covariant differential operator on $\bar{\mathcal{N}}_x$ with respect to $\breve{g}_1$. Garay states that [5] if the almost complex manifold $\mathcal{P}$ indicates

$$(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P})\mathcal{X}_a = 0 \tag{3}$$

for any $\mathcal{T}\bar{\mathcal{N}}_x$, then an almost paracomplex metric manifold is called nearly para-Kaehler structure. Equation (3) is equivalent to

$$(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P})\mathcal{Y}_b + (\bar{\nabla}_{\mathcal{Y}_b}\mathcal{P})\mathcal{X}_a = 0 \tag{4}$$

including any vector fields $\mathcal{X}_a, \mathcal{Y}_b$ on $\bar{\mathcal{N}}_x$.

Currently, let $\mathcal{N}_x$ be a submanifold of $(\mathcal{P}, \breve{g}_1)$. The Gauss and Weingarten equations are dedicated by

$$\bar{\nabla}_{\mathcal{X}_a}\mathcal{Y}_b = \nabla_{\mathcal{X}_a}\mathcal{Y}_b + h_1(\mathcal{X}_a, \mathcal{Y}_b), \tag{5}$$

$$\bar{\nabla}_{\mathcal{X}_a}N = -A_N\mathcal{X}_a + \nabla^{\perp}_{\mathcal{X}_a}N, \tag{6}$$

including $\mathcal{X}_a, \mathcal{Y}_b \in \Gamma(\mathcal{T}\mathcal{N}_x)$ and $N \in \Gamma(\mathcal{T}\mathcal{N}_b^{\perp})$, where h_1 is a second fundamental form of $\mathcal{N}_x$, A_N is the Weingarten endomorphism connected with N, and $\nabla^{\perp}$ is the normal connection. A_N and h_1 are related by

$$\breve{g}_1(A_N\mathcal{X}_a, \mathcal{Y}_b) = \breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Y}_b), N), \tag{7}$$

here $\breve{g}_1$ designates the semi-Riemannian metric on $\mathcal{N}_x$ with the one introduced on $\mathcal{N}_x$. For all tangent vector field $\mathcal{X}_a$, we denote

$$\mathcal{P}\mathcal{X}_a = R\mathcal{X}_a + S\mathcal{X}_a, \tag{8}$$

where $R\mathcal{X}_a$ is the tangential component of $\mathcal{P}\mathcal{X}_a$ and $S\mathcal{X}_a$ is the normal one. For all normal vector field N,

$$\mathcal{P}N = rN + sN, \tag{9}$$

where rN and sN are the tangential and normal components of $\mathcal{P}N$.

Now, denote by $\mathcal{G}_{\mathcal{X}_a}\mathcal{Y}_b$ and $\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b$ the tangential and normal parts of $(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P})\mathcal{Y}_b$, i.e.,

$$(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P})\mathcal{Y}_b = \mathcal{G}_{\mathcal{X}_a}\mathcal{Y}_b + \mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b, \tag{10}$$

for all $\mathcal{X}_a, \mathcal{Y}_b \in \Gamma(\mathcal{T}\mathcal{N}_x)$. For the proporties of $\mathcal{G}$ and $\mathcal{U}$, we refer to [8], which we express here for later use.

$$(m_1) \quad (a) \quad \mathcal{G}_{\mathcal{X}_a+\mathcal{Y}_b}\mathcal{W}_c = \mathcal{G}_{\mathcal{X}_a}\mathcal{W}_c + \mathcal{G}_{\mathcal{Y}_b}\mathcal{W}_c$$
$$(b) \quad \mathcal{U}_{\mathcal{X}_a+\mathcal{Y}_b}\mathcal{W}_c = \mathcal{U}_{\mathcal{X}_a}\mathcal{W}_c + \mathcal{U}_{\mathcal{Y}_b}\mathcal{W}_c$$

$$(m_2) \quad (a) \quad \mathcal{G}_{\mathcal{X}_a}(\mathcal{Y}_b + \mathcal{W}_c) = \mathcal{G}_{\mathcal{X}_a}\mathcal{Y}_b + \mathcal{G}_{\mathcal{X}_a}\mathcal{W}_c$$
$$(b) \quad \mathcal{U}_{\mathcal{X}_a}(\mathcal{Y}_b + \mathcal{W}_c) = \mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b + \mathcal{U}_{\mathcal{X}_a}\mathcal{W}_c$$

$$(m_3) \quad (a) \quad \breve{g}_1(\mathcal{G}_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{W}_c) = -\breve{g}_1(\mathcal{Y}_b, \mathcal{A}_{\mathcal{X}_a}\mathcal{W}_c)$$
$$(b) \quad \breve{g}_1(\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{V}_f) = -\breve{g}_1(\mathcal{Y}_b, \mathcal{G}_{\mathcal{X}_a}\mathcal{V}_f)$$

$$(m_4) \quad \mathcal{G}_{\mathcal{X}_a}\mathcal{P}\mathcal{Y}_b + \mathcal{U}_{\mathcal{X}_a}\mathcal{P}\mathcal{Y}_b = -\mathcal{P}(\mathcal{G}_{\mathcal{X}_a}\mathcal{Y}_b + \mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b)$$

for all $\mathcal{X}_a, \mathcal{Y}_b, \mathcal{W}_c \in \Gamma(\mathcal{T}\mathcal{N}_x)$ and $V_f \in \Gamma(\mathcal{T}\mathcal{N}_b^{\perp})$

On a semi-Riemannian submanifold $\mathcal{N}_x$ whose ambient space are nearly para-Kaehler manifold $\bar{\mathcal{N}_x}$. by equations (3) and (10), we get

$$(a)\mathcal{G}_{\mathcal{X}_a}\mathcal{Y}_b + \mathcal{G}_{\mathcal{Y}_b}\mathcal{X}_a = 0 \quad (b)\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b + \mathcal{U}_{\mathcal{Y}_b}\mathcal{X}_a = 0 \tag{11}$$

for any $\mathcal{X}_a, \mathcal{Y}_b \in \Gamma(\mathcal{T}\mathcal{N}_x)$

The mean curvature vector is indicated by

$$H = \frac{1}{n} trace h_1. \tag{12}$$

Definition 2.1 *We call that a submanifold $\mathcal{N}_x$ whose ambient spaces are nearly para-Kaehler manifold $(\bar{\mathcal{N}_x}, \mathcal{P}, \breve{g}_1)$ is pointwise slant if for all time-like or space-like tangent vector field $\mathcal{X}_a$, the ratio $\breve{g}_1(R\mathcal{X}_a, R\mathcal{X}_a)/\breve{g}_1(\mathcal{P}\mathcal{X}_a, P\mathcal{X}_a)$ is nonconstant. Moreover, a submanifold $\mathcal{N}_x$ whose ambient spaces are nearly para-Kaehler manifold $\bar{\mathcal{N}_x}$ is called pointwise slant [2] if, at each point $\mathtt{p} \in \mathcal{N}_x$, the Wirtinger angle $\theta(X)$ between $\mathcal{P}\mathcal{X}_a$ and $\mathcal{T}_p\mathcal{N}_x$ is dependent on the choice of the nonzero $\mathcal{X}_a \in \mathcal{T}_p\mathcal{N}_x$. In this instance, the Wirtinger angle causes a real-valued function $\theta : \mathcal{T}\mathcal{N}_x - 0 \to R$, which is called the slant function or Wirtinger function of the pointwise slant submanifold.*

We express that a pointwise slant submanifold of nearly para-Kaehler manifold is called slant [4] if its Wirtinger function θ is globally constant.

If $\mathcal{N}_x$ is a paracomplex (para-holomorphic) submanifold, in that case, $\mathcal{P}\mathcal{X}_a = R\mathcal{X}_a$ and the above ratio is equal to 1. Moreover, if $\mathcal{N}_x$ is totally real (antiinvariant), then $R = 0$, so $\mathcal{P}\mathcal{X}_a = S\mathcal{X}_a$ and the above ratio equals 0. Hence, both paracomplex submanifolds and totally real are the special situations of pointwise slant submanifolds. Neither totally real nor paracomplex pointwise slant submanifold can be called a proper pointwise slant.

Definition 2.2 *Let $\mathcal{N}_x$ be a proper pointwise slant submanifold whose ambient spaces are para-Hermitian manifold $(\bar{\mathcal{N}_x}, \mathcal{P}, \breve{g}_1)$. We call it of* *type-*1 *if for any spacelike(timelike) vector field* $\mathcal{X}_a$*,* $R\mathcal{X}_a$ *is timelike(spacelike), also* $\frac{|R\mathcal{X}_a|}{|\mathcal{P}\mathcal{X}_a|} > 1$*.*

*type-*2 *if for any spacelike(timelike) vector field* $\mathcal{X}_a$*,* $R\mathcal{X}_a$ *is timelike(spacelike), also* $\frac{|R\mathcal{X}_a|}{|\mathcal{P}\mathcal{X}_a|} < 1$*.*

Similar to the approach used by Alegra and Carriazo, the following theorem and results were obtained [2].

Theorem 2.3 *Let* $\mathcal{N}_x$ *be a pointwise slant submanifold of an almost paracomplex metric manifold* $(\bar{\mathcal{N}}_x, \mathcal{P}, \breve{g}_1)$*. So,*

(a) $\mathcal{N}_x$ *is the pointwise slant submanifold of type-1 if and only if for any spacelike (timelike) vector field* $\mathcal{X}_a$*,* $R\mathcal{X}_a$ *is timelike (spacelike), also there arises function* $\mu \in (1, +\infty)$*. Therefore,*

$$\mu = R^2 = cosh^2\theta. \tag{13}$$

θ *denotes the slant function of* N_x*.*

(b) $\mathcal{N}_x$ *is pointwise slant submanifold of type-*2 *if and only if for any spacelike (timelike) vector field* $\mathcal{X}_a$*,* $R\mathcal{X}_a$ *is timelike (spacelike), also there arises a function* $\mu \in (0, 1)$*. Therefore,*

$$\mu = R^2 = cos^2\theta, \tag{14}$$

here θ *indicates the slant function of* N_x*.*

Proof. Firstly, if $\mathcal{N}_x$ is a pointwise slant submanifold of type-1 for any spacelike tangent vector field $\mathcal{X}_a$, $R\mathcal{X}_a$ is timelike, and by the equation (1), $\mathcal{P}\mathcal{X}_a$ is also timelike. Furthermore, they supply $|R\mathcal{X}_a|/|\mathcal{P}\mathcal{X}_a| > 1$, so there exists the slant function α. This follows from

$$\cosh\theta = \frac{|R\mathcal{X}_a|}{|\mathcal{P}\mathcal{X}_a|} = \frac{\sqrt{-\breve{g}_1(R\mathcal{X}_a, R\mathcal{X}_a))}}{\sqrt{-\breve{g}_1(\mathcal{P}\mathcal{X}_a, \mathcal{P}\mathcal{X}_a)}} \tag{15}$$

Using (1) and (17) , we have

$$\breve{g}_1(R^2\mathcal{X}_a, \mathcal{X}_a) = \cosh^2\theta\breve{g}_1(\mathcal{X}_a, \mathcal{X}_a).$$

Thus, we get $R^2\mathcal{X}_a = cosh^2\theta(\mathcal{X}_a)$, so $\mu = R^2 = \cosh^2\theta$.

We can study anything using the same method for any time-like tangent vector field $\mathcal{Z}_d$. Currently, $R\mathcal{Z}_d$ and $\mathcal{P}\mathcal{Z}_d$ are spacelike. Therefore, in place of (15), we get

$$\cosh\theta = \frac{|R\mathcal{Z}_d|}{|\mathcal{P}\mathcal{Z}_d|} = \frac{\sqrt{\breve{g}_1(R\mathcal{Z}_d, R\mathcal{Z}_d))}}{\sqrt{\breve{g}_1(\mathcal{P}\mathcal{Z}_d, \mathcal{P}\mathcal{Z}_d)}}$$

Since $R^2\mathcal{X}_a = cosh^2\theta(\mathcal{X}_a)$, for any spacelike and timelike $\mathcal{X}_a$, it further provides for lightlike vector fields. Therefore, we get $\mu = R^2 = \cosh^2\theta$. The contrary is (a) direct calculation.

In a similar way, we have $(b)^{\cdot}$ Lastly, for both pointwise slant submanifolds of type-*1* and type-2, if $\mathcal{X}_a$ is spacelike, $\mathcal{P}\mathcal{X}_a$ is timelike. Thus, all pointwise slant submanifolds of type-*1* and type-2 should be a semi-Riemann manifold.

Using (1),(5),(13), and (14), we get

Corollary 2.4 Let $\mathcal{N}_x$ be a pointwise slant submanifold of almost paracomplex metric manifold $(\bar{\mathcal{N}}_x, \mathcal{P}, \breve{g}_1)$ with the slant function θ. Later, for any nonnull vector fields $\mathcal{X}_a, \mathcal{Y}_b \in \mathcal{T}\mathcal{N}_x$, we obtain:
If $\mathcal{N}_x$ is of type-1, then

$$\begin{aligned} \breve{g}_1(R\mathcal{X}_a, R\mathcal{Y}_b) &= -\cosh^2\theta \breve{g}_1(\mathcal{X}_a, \mathcal{Y}_b) \\ \breve{g}_1(S\mathcal{X}_a, S\mathcal{Y}_b) &= \sinh^2\theta \breve{g}_1(\mathcal{X}_a, \mathcal{Y}_b)). \end{aligned} \tag{16}$$

If $\mathcal{N}$ is of type-2, then

$$\begin{aligned} \breve{g}_1(R\mathcal{X}_a, R\mathcal{Y}_b) &= -\cos^2\theta \breve{g}_1(\mathcal{X}_a, \mathcal{Y}_b), \\ \breve{g}_1(S\mathcal{X}_a, S\mathcal{Y}_b) &= -\sin^2\theta \breve{g}_1(\mathcal{X}_a, \mathcal{Y}_b). \end{aligned} \tag{17}$$

Using (1),(5),(6),(13), and (14), we have

Corollary 2.5 Let $\mathcal{N}_x$ be a pointwise slant submanifold of an almost paracomplex metric manifold $(\bar{\mathcal{N}}_x, \mathcal{P}, \breve{g}_1)$. Later, let $\mathcal{N}_x$ be a pointwise slant submanifold of almost paracomplex metric manifold $\bar{\mathcal{N}}_x$. Therefore, $\mathcal{N}_x$ is a pointwise slant submanifold of
(for type-$\mathtt{1}$) if and only if

$$rS\mathcal{X}_a = -\sinh^2\theta\mathcal{X}_a \quad and \quad sS\mathcal{X}_a = -RS\mathcal{X}_a \tag{18}$$

(for type-$\mathtt{2}$) if and only if

$$rS\mathcal{X}_a = \sin^2\theta\mathcal{X}_a \quad and \quad sS\mathcal{X}_a = -RS\mathcal{X}_a \tag{19}$$

for all timelike (spacelike) vector field $\mathcal{X}_a$.

Let us consider almost paracomplex structure on $\bar{R}^6_3$:

$$P(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}, \ \ P(\frac{\partial}{\partial y_i}) = \frac{\partial}{\partial x_i}, \ \ \breve{g}_1 = (+,-,+,-,+,-)$$

and $\breve{g}_1$ is pseudo-Riemannian metric. Also, $(x_1, y_1, x_2, y_2, x_3, y_3)$ denotes the cartesian coordinates over $\bar{R}^6_3$. At the moment, we can give several examples with pointwise slant submanifolds.

Example 2.1 For $\mathtt{k}+\mathtt{n} > 0$, with

$$\psi(\mathtt{k}, \mathtt{n}, \mathtt{z}) = (cosh\mathtt{k}, \cosh\mathtt{n}, \sinh\mathtt{n}, \sinh\mathtt{k}, \mathtt{z}, \pi),$$

we have a pointwise slant submanifold of type-$\mathtt{1}$ in $(\bar{\mathcal{R}}^6_3, \mathcal{P}, \breve{g}_1)$, $\mu = \mathcal{R}^2 = cosh^2(k+n)$.

Example 2.2 For $\mathtt{a} \neq \mathtt{b}$, with

$$\psi(\mathtt{a}, \mathtt{b}, \mathtt{r}) = (sin\mathtt{a}, sin\mathtt{b}, cos\mathtt{a}, cos\mathtt{b}, \mathtt{r}, \mathtt{h}),$$

we have a pointwise slant submanifold with type-$\mathtt{2}$ in $(\bar{\mathcal{R}}^6_3, \mathcal{P}, \breve{g}_1)$, $\mu = \mathcal{R}^2 = cos^2(\mathtt{a}-\mathtt{b})$,
for $(\mathtt{a}-\mathtt{b}) \in (0, \frac{\pi}{2}))$.

Example 2.3 For $a \neq b \neq 0$ with,

$$\psi(a, b, \mathtt{v}) = (asin\mathtt{q}, absin\mathtt{q}, acos\mathtt{q}, abcos\mathtt{q}, \mathtt{v}, \mathtt{n})$$

describes a pointwise slant submanifold in $(\bar{\mathcal{R}}^6_3, \mathcal{P}, \breve{g}_1)$, with $\mu = \mathcal{R}^2 = \frac{1}{1-b^2}$ and it is
(i) of type-1 if $0 < 1 - b^2 < 1$,
(ii)of type-2 if $1 - b^2 > 1$.

3. Pointwise hemislant submanifolds whose ambient spaces are nearly para-Kaehler manifolds

Definition 3.1 *A semi-Riemannian submanifold $\mathcal{N}_x$ whose ambient spaces are nearly para-Kaehler manifold $(\bar{\mathcal{N}}_x, \mathcal{P}, \breve{g}_1)$ is called pointwise hemislant submanifold if there exist two orthogonal distributions $D^{\perp}_t$, D^{α}_n with $\mathcal{N}_x$. Therefore,*
1) *$\mathcal{T}\mathcal{N}_x = \mathcal{D}^{\perp}_t \oplus \mathcal{D}^{\alpha}_n$*
2) *The distribution $\mathcal{D}^{\perp}_t$ is a totally real distribution, $\mathcal{P}\mathcal{D}^{\perp}_t \subset \mathcal{T}^{\perp}\mathcal{N}_x$;*
3) *The distribution $\mathcal{D}^{\alpha}_n$ is a pointwise slant distribution.*
Then, we define the corner θ as the hemislant function with the pointwise hemislant submanifold $\mathcal{N}_x$. A pointwise hemislant submanifold $\mathcal{N}_x$ is called proper if its hemislant function specifies $\theta \neq 0, \frac{\pi}{2}$, and θ is not constant on $\mathcal{N}_x$.

Definition 3.2 *Let $\mathcal{N}_x$ be a proper pointwise hemislant submanifold of an almost paracomplex metric manifold $(\bar{\mathcal{N}}_x, \mathcal{P}, \breve{g}_1)$. Let $\mathcal{D}^{\alpha}_n$ be a pointwise slant distribution on $\mathcal{N}_x$. We sat that it is of*
type-1 if for any spacelike(timelike) vector field $\mathcal{X}_a$, $R\mathcal{X}_a$ is timelike(spacelike), and also $\frac{|R\mathcal{X}_a|}{|\mathcal{P}\mathcal{X}_a|} > 1$.
type-2 if for any spacelike(timelike) vector field $\mathcal{X}_a$, $R\mathcal{X}_a$ is timelike(spacelike), and also $\frac{|R\mathcal{X}_a|}{|\mathcal{P}\mathcal{X}_a|} < 1$.

Theorem 3.3 *Let $\mathcal{N}_x$ be a pointwise hemislant submanifold of almost paracomplex metric manifold $(\bar{\mathcal{N}}_x, \mathcal{P}, \breve{g}_1)$. Thus,*
(a) *$\mathcal{D}^{\alpha}_n$ is a pointwise slant distribution of type-1 if and only if for any spacelike (timelike) vector field $\mathcal{X}_a$, $R\mathcal{X}_a$ is timelike (spacelike), also there arises a function $\mu \in (1, +\infty)$. Therefore,*

$$\mu = R^2 = cosh^2\theta, \tag{20}$$

here θ defines the hemislant function of N_x.
(b) *$\mathcal{D}^{\alpha}_n$ is a pointwise slant distribution of type-2 if and only if for any spacelike (timelike) vector field $\mathcal{X}_a$, $R\mathcal{X}_a$ is timelike (spacelike), also there arises a function $\mu \in (0, 1)$. Therefore,*

$$\mu = R^2 = cos^2\theta, \tag{21}$$

here θ denotes the hemislant function of $\mathcal{N}_x$.

Proof. The proof is conducted similarly to the proof of (13) and (14),
By utilizing (1),(5), (20), and (21), we get

Corollary 3.4 Let $\mathcal{N}_x$ be a pointwise hemislant submanifold of almost paracomplex metric manifold $(\bar{\mathcal{N}}_x, \mathcal{P}, \breve{g}_1)$ with the hemislant function θ. For nonnull vector fields $\mathcal{X}_a, \mathcal{Y}_b \in \mathcal{T}\mathcal{N}_x$, if $\mathcal{D}_n^\alpha$ is of type-1, then we obtain

$$\breve{g}_1(R\mathcal{X}_a, R\mathcal{Y}_b) = -\cosh^2\theta\breve{g}_1(\mathcal{X}_a, \mathcal{Y}_b),$$
$$\breve{g}_1(S\mathcal{X}_a, S\mathcal{Y}_b) = \sinh^2\theta\breve{g}_1(\mathcal{X}_a, \mathcal{Y}_b). \tag{22}$$

If $\mathcal{D}_n^\alpha$ is of type-2, then we have

$$\breve{g}_1(R\mathcal{X}_a, R\mathcal{Y}_b) = -\cos^2\theta\breve{g}_1(\mathcal{X}_a, \mathcal{Y}_b),$$
$$\breve{g}_1(S\mathcal{X}_a, S\mathcal{Y}_b) = -\sin^2\theta\breve{g}_1(\mathcal{X}_a, \mathcal{Y}_b). \tag{23}$$

A hemislant submanifold $\mathcal{N}_x$ of an almost paracomplex metric manifold $\bar{\mathcal{N}}_x$ is called mixed geodesic if $h_1(\mathcal{X}_a, \mathcal{Z}_d) = 0$, for $\mathcal{X}_a \in \mathcal{D}_n^\alpha$ and $\mathcal{Z}_d \in \mathcal{D}_t^\perp$

Let $\mathcal{N}_x$ be a pointwise hemislant submanifold whose ambient spaces are nearly para-Kaehler manifold $\bar{\mathcal{N}}_x$. The normal bundle $\mathcal{T}^\perp N_x$ can be decomposed as $\mathcal{T}^\perp N_x = \mathcal{P}D_t^\perp \oplus SD_n^\alpha \oplus \mu$.
Now, we get some results for use in the next section.

Lemma 3.5. *Let $\mathcal{N}_x$ be a proper pointwise hemislant type-1 and type-2 submanifold whose ambient spaces are nearly para-Kaehler manifold $(\bar{\mathcal{N}}_x, \mathcal{P}, \breve{g}_1)$. In that case,*
1) For type-1,

$$\begin{aligned}\breve{g}_1(\nabla_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{Z}_d) &= sech^2\theta(\breve{g}_1(A_{SR\mathcal{Y}_b}\mathcal{X}_a, \mathcal{Z}_d) - \breve{g}_1(A_{\mathcal{P}\mathcal{Z}_d}\mathcal{X}_a, R\mathcal{Y}_b) \\ &+ \breve{g}_1(\mathcal{U}_{\mathcal{X}_a}\mathcal{Z}_d, S\mathcal{Y}_b) + \breve{g}_1(\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{P}\mathcal{Z}_d))\end{aligned} \tag{24}$$

2) For type-2,

$$\begin{aligned}\breve{g}_1(\nabla_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{Z}_d) &= sec^2\theta(\breve{g}_1(A_{SR\mathcal{Y}_b}\mathcal{X}_a, \mathcal{Z}_d) - \breve{g}_1(A_{\mathcal{P}\mathcal{Z}_d}\mathcal{X}_a, R\mathcal{Y}_b) \\ &+ \breve{g}_1(\mathcal{U}_{\mathcal{X}_a}\mathcal{Z}_d, S\mathcal{Y}_b) + \breve{g}_1(\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{P}\mathcal{Z}_d))\end{aligned} \tag{25}$$

for nonnull vector fields $\mathcal{X}_a, \mathcal{Y}_b \in \mathcal{D}_n^\alpha$ and $\mathcal{Z}_d \in \mathcal{D}_t^\perp$.

Proof. Using (1),(2),(5),(6),(7),(8),(9),(10), and (18), we write

$$\begin{aligned}
\breve{g}_1(\nabla_{\mathcal{X}_a}\mathcal{Y}_b,\mathcal{Z}_d) &= \breve{g}_1(\bar{\nabla}_{\mathcal{X}a}\mathcal{Y}_b,\mathcal{Z}_d)\\
&= -\breve{g}_1(\mathcal{P}\bar{\nabla}_{\mathcal{X}a}\mathcal{Y}_b,\mathcal{P}\mathcal{Z}_d)\\
&= -\breve{g}_1(\bar{\nabla}_{\mathcal{X}a}\mathcal{P}\mathcal{Y}_b,\mathcal{P}\mathcal{Z}_d)+\breve{g}_1((\bar{\nabla}_{\mathcal{X}a}\mathcal{P})\mathcal{Y}_b,\mathcal{P}\mathcal{Z}_d)\\
&= -\breve{g}_1(\bar{\nabla}_{\mathcal{X}a}\mathcal{R}\mathcal{Y}_b,\mathcal{P}\mathcal{Z}_d)+\breve{g}_1(\bar{\nabla}_{\mathcal{X}a}\mathcal{S}\mathcal{Y}_b,\mathcal{P}\mathcal{Z}_d)+\breve{g}_1(\mathcal{U}_{\mathcal{X}a}\mathcal{Y}_b,\mathcal{P}\mathcal{Z}_d)\\
&= -\breve{g}_1(h_1(\mathcal{X}_a,R\mathcal{Y}_b),\mathcal{P}\mathcal{Z}_d)+\breve{g}_1(\mathcal{P}\bar{\nabla}_{\mathcal{X}a}S\mathcal{Y}_b,\mathcal{Z}_d)+\breve{g}_1(\mathcal{U}_{\mathcal{X}a}\mathcal{Y}_b,\mathcal{P}\mathcal{Z}_d)\\
&= -\breve{g}_1(h_1(\mathcal{X}_a,R\mathcal{Y}_b),\mathcal{P}\mathcal{Z}_d)+\breve{g}_1(\bar{\nabla}_{\mathcal{X}a}\mathcal{P}S\mathcal{Y}_b,\mathcal{Z}_d)\\
&- \breve{g}_1((\bar{\nabla}_{\mathcal{X}a}\mathcal{P})S\mathcal{Y}_b,\mathcal{Z}_d)+\breve{g}_1(\mathcal{U}_{\mathcal{X}a}\mathcal{Y}_b,\mathcal{P}\mathcal{Z}_d)\\
&= -\breve{g}_1(h_1(\mathcal{X}_a,R\mathcal{Y}_b),\mathcal{P}\mathcal{Z}_d)+\breve{g}_1(\bar{\nabla}_{\mathcal{X}a}rS\mathcal{Y}_b,\mathcal{Z}_d)\\
&- \breve{g}_1(\bar{\nabla}_{\mathcal{X}a}sS\mathcal{Y}_b,\mathcal{Z}_d)-\breve{g}_1(\mathcal{G}_{\mathcal{X}a}\mathcal{S}\mathcal{Y}_b,\mathcal{Z}_d)+\breve{g}_1(\mathcal{U}_{\mathcal{X}a}\mathcal{Y}_b,\mathcal{P}\mathcal{Z}_d)\\
&= -\breve{g}_1(h_1(\mathcal{X}_a,R\mathcal{Y}_b),\mathcal{P}\mathcal{Z}_d)-2sinh\theta cosh\theta\mathcal{X}_a(\theta)\breve{g}_1(\mathcal{Y}_b,\mathcal{Z}_d)\\
&- sinh^2\theta\breve{g}_1(\bar{\nabla}_{\mathcal{X}a}\mathcal{Y}_b,\mathcal{Z}_d)-\breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}SR\mathcal{Y}_b,\mathcal{Z}_d)\\
&+ \breve{g}_1(\mathcal{U}_{\mathcal{X}_a}\mathcal{Z}_d,\mathcal{S}\mathcal{Y}_b)+\breve{g}_1(\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b,\mathcal{P}\mathcal{Z}_d)\\
&= -\breve{g}_1(h_1(\mathcal{X}_a,R\mathcal{Y}_b),\mathcal{P}\mathcal{Z}_d)-sinh^2\theta\breve{g}_1(\bar{\nabla}_{\mathcal{X}a}\mathcal{Y}_b,\mathcal{Z}_d)\\
&+ \breve{g}_1(A_{SR\mathcal{Y}_b}\mathcal{X}_a,\mathcal{Z}_d)-\breve{g}_1(\nabla^{\perp}_{\mathcal{X}_a}SR\mathcal{Y}_b,\mathcal{Z}_d)\\
&+ \breve{g}_1(\mathcal{U}_{\mathcal{X}_a}\mathcal{Z}_d,\mathcal{S}\mathcal{Y}_b)+\breve{g}_1(\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b,\mathcal{P}\mathcal{Z}_d)
\end{aligned}$$

Utilizing (5) and (7), we have

$$\begin{aligned}
cosh^2\theta\breve{g}_1(\nabla_{\mathcal{X}_a}\mathcal{Y}_b,\mathcal{Z}_d) &= \breve{g}_1(A_{SR\mathcal{Y}_b}\mathcal{X}_a,\mathcal{Z}_d)-\breve{g}_1(A_{\mathcal{P}\mathcal{Z}_d}\mathcal{X}_a,R\mathcal{Y}_b)\\
&+ \breve{g}_1(\mathcal{U}_{\mathcal{X}_a}\mathcal{Z}_d,S\mathcal{Y}_b)+\breve{g}_1(\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b,\mathcal{P}\mathcal{Z}_d)
\end{aligned}$$

This proves Case (1). In a similar way, the proof of Case (2) is obtained.
Now using the above lemma, we obtain

Corollary 3.6 Let $\mathcal{N}_x$ be a proper pointwise hemislant type1-2 submanifold whose ambient spaces are nearly para-Kaehler manifold $(\bar{\mathcal{N}}_x,\mathcal{P},\breve{g}_1)$. In that case, the proper pointwise slant distribution $\mathcal{D}_n^{\alpha}$ describes a totally geodesic foliation, if and only if

$$A_{SR\mathcal{X}_a}\mathcal{Z}_d-A_{\mathcal{P}\mathcal{Z}_d}R\mathcal{X}_a\in\Gamma(\mathcal{D}_t^{\perp})\quad \mathcal{U}_{\mathcal{X}_a}\mathcal{H}\in\Gamma(\mu) \tag{26}$$

for nonnull vector fields $\mathcal{Z}_d,\mathcal{W}_c\in\mathcal{D}_t^{\perp}$, $\mathcal{X}_a\in\mathcal{D}_n^{\alpha}$ and $\mathcal{H}\in\Gamma(\mathcal{T}N_x)$

Let us consider nearly para-Kaehler structure on $\bar{R}^6_3$:

$$\mathcal{P}(\frac{\partial}{\partial x_i})=\frac{\partial}{\partial y_i},\ \ \mathcal{P}(\frac{\partial}{\partial y_i})=\frac{\partial}{\partial x_i}$$

$\breve{g}_1=(+,-,+,-,+,-)$. $\breve{g}_1$ is pseudo-Riemannian metric. Also, $(x_1,\mathrm{y}_1,x_2,\mathrm{y}_2,x_3,\mathrm{y}_3)$ denotes the cartesian coordinates over $\bar{R}^6_3$. Then $(\bar{R}^6_3,\mathcal{P},\breve{g}_1)$ is a nearly para-Kaehler manifold.
Now, we will write some examples.

Let $\mathcal{N}_x$ be a semi-Riemannian submanifold of $\bar{\mathcal{R}}_3^6$ described by $\psi : \mathcal{N}_x \to \bar{\mathcal{R}}_3^6$.

Example 3.1 The semi-Riemannian submanifold $\mathcal{N}_x$ of $(\bar{\mathcal{R}}_3^6, \mathcal{P}, \breve{g}_1)$ nearly para-Kaehler manifold described by

$$\psi(m,n,k) = (sinhm, \sinh n, \cosh n, \cosh m, sinhk, \pi)$$

is a pointwise hemislant submanifold of type-1. Actually, the antiinvariant distribution is $\mathcal{D}_t^{\perp} = span\{coshk\frac{\partial}{\partial x_3}\}$ and pointwise slant distribution is $\mathcal{D}_n^{\alpha} = span\{\cosh m\frac{\partial}{\partial x_1} + \sinh m\frac{\partial}{\partial y_2}, \cosh n\frac{\partial}{\partial y_1} + sinhn\frac{\partial}{\partial x_2}\}$ and $R^2 = \cosh^2(m+n)(\mathcal{X})$ with $m+n>0$.

Example 3.2 The semi-Riemannian submanifold $\mathcal{N}_x$ of $(\bar{R}_3^6, \mathcal{P}, \breve{g}_1)$ nearly para-Kaehler manifold defined by

$$\psi(\mathtt{m}, \mathtt{n}, k) = (sin\mathtt{m}, sin\mathtt{n}, cos\mathtt{m}, cos\mathtt{n}, k, e),$$

is a pointwise hemislant submanifold of type-2. Actually, the distributions are $\mathcal{D}_t^{\perp} = span\{\frac{\partial}{\partial x_3}\}$ and $\mathcal{D}_n^{\alpha} = span\{cos\mathtt{m}\frac{\partial}{\partial x_1} - sin\mathtt{m}\frac{\partial}{\partial x_2}, cos\mathtt{n}\frac{\partial}{\partial y_1} - sinn\frac{\partial}{\partial y_2}\}$ with $R^2 = \cos^2(m-n)(\mathcal{X})$, $m-n \in (0, \frac{\pi}{2})$.

4. Warped product pointwise hemislant submanifolds whose ambient spaces are nearly para-Kaehler manifolds

Warped products are $\mathcal{N}_a$ and $\mathcal{N}_b$ two semi-Riemannian manifolds with metrics $\breve{g}_a$ and $\breve{g}_b$ also differentiable function k on $\mathcal{N}_a$. Projections of $\mathcal{N}_a \times \mathcal{N}_b$ are $\beta_1 : \mathcal{N}_a \times \mathcal{N}_b \to \mathcal{N}_a$ and $\beta_2 : \mathcal{N}_a \times \mathcal{N}_b \to \mathcal{N}_b$. Warped product manifold $\mathcal{N}_x = \mathcal{N}_a \times_k \mathcal{N}_b$ is the semi-Riemannian manifold $\mathcal{N}_a \times \mathcal{N}_b = (\mathcal{N}_a \times \mathcal{N}_b, \breve{g})$ with the semi-Riemannian structure; therefore,

$$\breve{g}(\mathcal{X}_a, \mathcal{Y}_b) = \breve{g}_1(\beta_{1*}\mathcal{X}_a, \beta_{1*}\mathcal{Y}_b) + (k \circ \beta_1)^2 \breve{g}_1(\beta_{2*}\mathcal{X}_a, \beta_{2*}\mathcal{Y}_b)$$

for every vector fields $\mathcal{X}_a, \mathcal{Y}_b \in \Gamma(T\mathcal{N}_x)$, where * indicates the tangent map. The function k is called the warping function. If the warping function is constant, the structure $\mathcal{N}_x$ is called trivial. However, if the warping function is nonconstant, the structure $\mathcal{N}_x$ is called nontrivial. $\mathcal{N}_a$ is totally geodesic and $\mathcal{N}_b$ is totally umbilical in $\mathcal{N}_x$ [12].

Let $\mathcal{N}_x = \mathcal{N}_a \times_k \mathcal{N}_b$ be a warped product manifold with the warping function k; therefore,

$$\nabla_{\mathcal{X}_a}\mathcal{Z}_d = \nabla_{\mathcal{Z}_d}\mathcal{X}_a = (\mathcal{X}_a lnk)\mathcal{Z}_d \tag{27}$$

for nonnull vector fields $\mathcal{X}_a \in \mathcal{T}\mathcal{N}_a$ and $\mathcal{Z}_d \in \mathcal{T}\mathcal{N}_b$ [3], where ∇ defines the Levi-Civita connections on $\mathcal{N}_x$. Also, as a result, we get

$$||grad(k)||^2 = \sum_{v=1}^{s}(e_v(k))^2 \tag{28}$$

for an orthonormal frame $(e_1, ..., e_s)$ on $\mathcal{N}_a$.
Here $grad(k)$ is the gradient of k (∇k) and k is a nonconstant function.

S. Uddin and others demonstrated the nonexistence in warped product hemislant submanifolds of $\mathcal{N}_x = \mathcal{N}_b^{\perp} \times_k \mathcal{N}_a^{\theta}$ in nearly Kaehler manifold [11]. In a similar way, we establish the nonexistence of nontrivial warped product hemislant submanifolds of $\mathcal{N}_x = \mathcal{N}_b^{\perp} \times_k \mathcal{N}_a^{\theta}$ in nearly para-Kaehler manifolds.

Theorem 4.1. *There do not exist nontrivial warped product pointwise hemislant submanifolds* $\mathcal{N}_x = \mathcal{N}_b^{\perp} \times_k \mathcal{N}_a^{\theta}$ *whose ambient spaces are nearly para-Kaehler manifolds* $\bar{\mathcal{N}}_x$*, where* $\mathcal{N}_b^{\perp}$ *is antiinvariant and* $\mathcal{N}_a^{\theta}$ *is a pointwise slant submanifold in* $\bar{\mathcal{N}}_x$*.*

Proof. The nonexistence of nontrivial warped product pointwise hemislant submanifolds $\mathcal{N}_x = \mathcal{N}_b^{\perp} \times_k \mathcal{N}_a^{\theta}$ of Kaehler manifolds was proved by Sahin from Theorem 4.2. of [9]. Similarly, we demonstrate the nonexistence of nontrivial warped product pointwise hemislant submanifolds $\mathcal{N}_x = \mathcal{N}_b^{\perp} \times_k \mathcal{N}_a^{\theta}$ of nearly para-Kaehler manifolds. Let us suppose that $\mathcal{N}_x$ is a warped product pointwise hemislant submanifold $\mathcal{N}_x = \mathcal{N}_b^{\perp} \times_k \mathcal{N}_a^{\theta}$ in $\bar{\mathcal{N}}_x$ so that $\mathcal{N}_b^{\perp}$ is a totally real submanifold and $\mathcal{N}_a^{\theta}$ is proper pointwise slant submanifold of $\bar{\mathcal{N}}_x$. In that case from (5), we get $\nabla_{\mathcal{X}_a}\mathcal{Y}_b = \bar{\nabla}_{\mathcal{X}_a}\mathcal{Y}_b$ for $\mathcal{X}_a \in \Gamma(\mathcal{T}\mathcal{N}_a^{\theta})$ and $\mathcal{Z}_d \in \Gamma(\mathcal{T}\mathcal{N}_b^{\perp})$

Using (27), we get $\mathcal{Z}_d(lnk)\breve{g}_1(\mathcal{X}_a, \mathcal{X}_a) = \breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}\mathcal{Z}_d, \mathcal{X}_a)$.

Since $\mathcal{X}_a$ and $\mathcal{Z}_d$ are orthogonal, we obtain $\mathcal{Z}_d(lnk)\breve{g}_1(\mathcal{X}_a, \mathcal{X}_a) = -\breve{g}_1(\mathcal{Z}_d, \bar{\nabla}_{\mathcal{X}_a}\mathcal{X}_a)$. Then from (1),(2),(3) and (5), we derive

$$\begin{aligned}
\mathcal{Z}_d(lnk)\breve{g}_1(\mathcal{X}_a, \mathcal{X}_a) &= \breve{g}_1(\mathcal{P}\mathcal{Z}_d, \mathcal{P}\bar{\nabla}_{\mathcal{X}_a}\mathcal{X}_a) \\
&= \breve{g}_1(\mathcal{P}\mathcal{Z}_d, \bar{\nabla}_{\mathcal{X}_a}\mathcal{P}\mathcal{X}_a - (\bar{\nabla}_{\mathcal{X}_a}\mathcal{P})\mathcal{X}_a) \\
&= \breve{g}_1(\mathcal{P}\mathcal{Z}_d, \bar{\nabla}_{\mathcal{X}_a}\mathcal{P}\mathcal{X}_a) \\
&= \breve{g}_1(\mathcal{P}\mathcal{Z}_d, h_1(\mathcal{X}_a, \mathcal{P}\mathcal{X}_a))
\end{aligned}$$

for $\mathcal{X}_a \in \Gamma\mathcal{T}(\mathcal{N}_a^{\theta})$ and $\mathcal{Z}_d \in \Gamma(\mathcal{T}\mathcal{N}_b^{\perp})$ Then, from (7), we derive

$$\breve{g}_1(A_{\mathcal{P}\mathcal{Z}_d}\mathcal{X}_a, \mathcal{P}\mathcal{X}_a) = -\breve{g}_1(\mathcal{P}\mathcal{Z}_d, \bar{\nabla}_{\mathcal{X}_a}\mathcal{P}\mathcal{X}_a)$$

Since $\bar{\nabla}$ is torsion-free $([\mathcal{X}_a, \mathcal{Y}_b] = \nabla_{\mathcal{X}_a}\mathcal{Y}_b + \nabla_{\mathcal{Y}_b}\mathcal{X}_a = 0)$, we derive

$$\breve{g}_1(A_{\mathcal{P}\mathcal{Z}_d}\mathcal{X}_a, \mathcal{P}\mathcal{X}_a) = -\breve{g}_1(\mathcal{P}\mathcal{Z}_d, [\mathcal{X}_a, \mathcal{P}\mathcal{X}_a] + \bar{\nabla}_{\mathcal{P}\mathcal{X}_a}\mathcal{X}_a)$$

Since $[\mathcal{X}_a, \mathcal{P}\mathcal{X}_a] \in \Gamma\mathcal{T}(\mathcal{N}_a^{\theta})$ and $\mathcal{P}\mathcal{Z}_d \in \Gamma(\mathcal{T}\mathcal{N}_b^{\perp})$, we derive

$$\breve{g}_1(A_{\mathcal{P}\mathcal{Z}_d}\mathcal{X}_a, \mathcal{P}\mathcal{X}_a) = -\breve{g}_1(\mathcal{P}\mathcal{Z}_d, \bar{\nabla}_{\mathcal{P}\mathcal{X}_a}\mathcal{X}_a)$$

Then from (1), we get

$$\breve{g}_1(A_{\mathcal{P}\mathcal{Z}_d}\mathcal{X}_a, \mathcal{P}\mathcal{X}_a) = -\breve{g}_1(\mathcal{P}\mathcal{Z}_d, \bar{\nabla}_{\mathcal{P}\mathcal{X}_a}\mathcal{P}^2\mathcal{X}_a)$$

From (3), we have

$$\breve{g}_1(A_{\mathcal{P}\mathcal{Z}_d}\mathcal{X}_a, \mathcal{P}\mathcal{X}_a) = -\breve{g}_1(\mathcal{P}\mathcal{Z}_d, (\bar{\nabla}_{\mathcal{P}\mathcal{X}_a}\mathcal{P})\mathcal{P}\mathcal{X}_a + \mathcal{P}\bar{\nabla}_{\mathcal{P}\mathcal{X}_a}\mathcal{P}\mathcal{X}_a)$$

$$\breve{g}_1(A_{\mathcal{P}\mathcal{Z}_d}\mathcal{X}_a, \mathcal{P}\mathcal{X}_a) = -\breve{g}_1(\mathcal{P}\mathcal{Z}_d, \mathcal{P}\bar{\nabla}_{\mathcal{P}\mathcal{X}_a}\mathcal{P}\mathcal{X}_a)$$

Using (1), we arrive

$$\breve{g}_1(A_{\mathcal{P}\mathcal{Z}_d}\mathcal{X}_a, \mathcal{P}\mathcal{X}_a) = \breve{g}_1(\mathcal{Z}_d, \bar{\nabla}_{\mathcal{P}\mathcal{X}_a}\mathcal{P}\mathcal{X}_a)$$

Since $\mathcal{P}\mathcal{X}_a$ and $\mathcal{Z}_d$ are orthogonal, we obtain

$$\breve{g}_1(A_{\mathcal{P}\mathcal{Z}_d}\mathcal{X}_a, \mathcal{P}\mathcal{X}_a) = -\breve{g}_1(\bar{\nabla}_{\mathcal{P}\mathcal{X}_a}\mathcal{Z}_d, \mathcal{P}\mathcal{X}_a)$$

Using (7) and (27), we get

$$\breve{g}_1(h_1(\mathcal{X}_a, \mathcal{P}\mathcal{X}_a)) = -\mathcal{Z}_d(lnk)\breve{g}_1(\mathcal{P}\mathcal{X}_a, \mathcal{P}\mathcal{X}_a)$$

Using (1) and (5), we get

$$\breve{g}_1(h_1(\mathcal{X}_a, \mathcal{P}\mathcal{X}_a)) = \mathcal{Z}_d(lnk)\breve{g}_1(\mathcal{X}_a, \mathcal{X}_a)$$

$$\breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P}\mathcal{X}_a - \nabla_{\mathcal{X}_a}\mathcal{P}\mathcal{X}_a, \mathcal{P}\mathcal{Z}_d) = \mathcal{Z}_d(lnk)\breve{g}_1(\mathcal{X}_a, \mathcal{X}_a)$$

$$-\breve{g}_1(\nabla_{\mathcal{X}_a}\mathcal{P}\mathcal{X}_a, \mathcal{P}\mathcal{Z}_d) = \mathcal{Z}_d(lnk)\breve{g}_1(\mathcal{X}_a, \mathcal{X}_a)$$

$$-\breve{g}_1(\mathcal{P}\nabla_{\mathcal{X}_a}\mathcal{X}_a, \mathcal{P}\mathcal{Z}_d) = \mathcal{Z}_d(lnk)\breve{g}_1(\mathcal{X}_a, \mathcal{X}_a)$$

$$\breve{g}_1(\nabla_{\mathcal{X}_a}\mathcal{X}_a, \mathcal{Z}_d) = \mathcal{Z}_d(lnk)\breve{g}_1(\mathcal{X}_a, \mathcal{X}_a)$$

Since $\mathcal{X}_a$ and $\mathcal{Z}_d$ are orthogonal, using (27), we obtain

$$-\breve{g}_1(\nabla_{\mathcal{X}_a}\mathcal{Z}_d, \mathcal{X}_a) = \mathcal{Z}_d(lnk)\breve{g}_1(\mathcal{X}_a, \mathcal{X}_a)$$

$$-\mathcal{Z}_d(lnk)\breve{g}_1(\mathcal{X}_a, \mathcal{X}_a) = \mathcal{Z}_d(lnk)\breve{g}_1(\mathcal{X}_a, \mathcal{X}_a)$$

Since $2\mathcal{Z}_d(lnk)\breve{g}_1(\mathcal{X}_a, \mathcal{X}_a) = 0$, k is constant. Therefore, the proof is complete.

Now, we write examples to demonstrate the existence of pointwise hemislant nontrivial warped product $\mathcal{N}_x = \mathcal{N}_a^\theta \times_k \mathcal{N}_b^\perp$ submanifolds of nearly para-Kaehler manifold.

Let $\mathcal{N}_x$ be a semi-Riemannian submanifold of $\bar{\mathcal{R}}_3^6$ described by $\psi : \mathcal{N}_x \to \bar{\mathcal{R}}_3^6$.

Example 4.1 For $m + n > 0$ and $\mathtt{m} + \mathtt{n} \in \mathcal{R}$, with

$$\psi(m, \mathtt{n}, \mathtt{c}) = (cosh\mathtt{m}, \cosh n, \sinh n, \sinh m, c^3, \alpha)$$

$$\psi_{\mathtt{m}} = \sinh \mathtt{m}\frac{\partial}{\partial x_1} + \cosh \mathtt{m}\frac{\partial}{\partial y_2}, \quad \psi_{\mathtt{n}} = \sinh \mathtt{n}\frac{\partial}{\partial y_1} + \cosh \mathtt{n}\frac{\partial}{\partial x_2}$$

$$\psi_c = +3c^2\frac{\partial}{\partial x_3}$$

Then we get

$$\mathcal{P}\psi_{\mathtt{m}} = \sinh \mathtt{m}\frac{\partial}{\partial y_1} + \cosh \mathtt{m}\frac{\partial}{\partial x_2}, \quad \mathcal{P}\psi_{\mathtt{n}} = \sinh \mathtt{n}\frac{\partial}{\partial x_1} + \cosh \mathtt{n}\frac{\partial}{\partial y_2}, \quad \mathcal{P}\psi_c = 3c^2\frac{\partial}{\partial y_3},$$

which describes a pointwise hemislant submanifold $\mathcal{N}_x^3$ with type-1 in $(\bar{\mathcal{R}}_3^6, \mathcal{P}, \breve{g}_1)$ nearly para-Kaehler manifold with $\mu = \mathcal{R}^2 = cosh^2(\mathtt{m}+\mathtt{n})$. Actually, $D_n^\alpha = span\{\psi_{\mathtt{m}}, \psi_{\mathtt{n}}\}$ is pointwise slant distribution with hemislant function and $\mathcal{D}_t^\perp = span\{\psi_c\}$ is antiinvariant distribution.
It is easy to notice that D_n^α and $\mathcal{D}_t^\perp$ distributions are integrable. The induced metric tensor $g_{\mathcal{N}_x}$ on $\mathcal{N}_x = \mathcal{N}_a^\theta \times_k \mathcal{N}_b^\perp$ is given by $g_{\mathcal{N}_x} = -{d_m}^2 + {d_n}^2 + (9c^4){d_c}^2 + {d_t}^2$
Thus, $\mathcal{N}_x$ is pointwise hemislant nontrivial warped product type-1 submanifold of $\bar{\mathcal{R}}_3^6$ nearly para-Kaehler manifold with warping function $k = 3c^2$.

Example 4.2 For $\mathtt{m}-\mathtt{n} \in (0, \frac{\pi}{2})$, with

$$\psi(\mathtt{m},\mathtt{n},c) = (cos\mathtt{m}, cos\mathtt{n}, sin\mathtt{m}, sin\mathtt{n}, sinc, \pi)$$

$$\psi_{\mathtt{m}} = -sin\mathtt{m}\frac{\partial}{\partial x_1} + cosm\frac{\partial}{\partial x_2}, \quad \psi_n = -\sin\mathtt{n}\frac{\partial}{\partial y_1} + cosn\frac{\partial}{\partial y_2}$$

$$\psi_c = \cos c\frac{\partial}{\partial x_3}$$

Then we get

$$\mathcal{P}\psi_{\mathtt{m}} = -\sin\mathtt{m}\frac{\partial}{\partial y_1} + \cos\mathtt{m}\frac{\partial}{\partial y_2}, \quad \mathcal{P}\psi_{\mathtt{n}} = -\sin\mathtt{n}\frac{\partial}{\partial x_1} + \cos\mathtt{n}\frac{\partial}{\partial x_2}, \quad \mathcal{P}\psi_c = \cos c\frac{\partial}{\partial y_3},$$

which describes a pointwise hemislant submanifold with type-2 in $(\bar{\mathcal{R}}_3^6, \mathcal{P}, \breve{g}_1)$ with $\mu = \mathcal{R}^2 = cos^2(\mathtt{m}-\mathtt{n})$. $D_n^\alpha = span\{\psi_{\mathtt{m}}, \psi_{\mathtt{n}}\}$ is pointwise slant distribution with hemislant function, $\mathcal{D}_t^\perp = span\{\psi_c\}$ is antiinvariant distribution, and $\mathcal{P}\psi_c \perp T\mathcal{N}_x = span\{\psi_{\mathtt{m}}, \psi_{\mathtt{n}}\}$.
It is easy to notice that D_n^α and $\mathcal{D}_t^\perp$ distributions are integrable. The induced metric tensor $g_{\mathcal{N}_x}$ on $\mathcal{N}_x = \mathcal{N}_a^\theta \times_k \mathcal{N}_b^\perp$ is given by $g_{\mathcal{N}_x} = {d_m}^2 - {d_n}^2 + (\cos^2 c){d_c}^2$.
Thus, $\mathcal{N}_x^3$ is pointwise hemislant nontrivial warped product type-2 submanifold of $\bar{\mathcal{R}}_3^6$ nearly para-Kaehler manifold with warping function $k = cosc$.
Now, we demonstrate lemmas in the below for later use.

Lemma 4.2 *Let $\mathcal{N}_x = \mathcal{N}_a^\theta \times_k \mathcal{N}_b^\perp$ be a warped product pointwise hemislant type1-2 submanifold whose ambient spaces are nearly para-Kaehler manifolds $\bar{\mathcal{N}}_x$, where $\mathcal{N}_b^\perp$ and $\mathcal{N}_a^\theta$ are totally real and proper pointwise slant submanifolds of $\bar{\mathcal{N}}_x$. In that case,*

$$-2\breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Y}_b), \mathcal{P}\mathcal{Z}_d) = \breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Z}_d), S\mathcal{Y}_b) + \breve{g}_1(h_1(\mathcal{Y}_b, \mathcal{Z}_d), S\mathcal{X}_a) \tag{29}$$

For any $\mathcal{X}_a, \mathcal{Y}_b \in T\mathcal{N}_a^\theta$ and $\mathcal{Z}_d \in T\mathcal{N}_b^\perp$.

Proof. Using (3),(5),(6), (7), and (8), we get

$$\breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Z}_d), \mathcal{S}\mathcal{Y}_b) = \breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}\mathcal{Z}_d, \mathcal{P}\mathcal{Y}_b) - \breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}\mathcal{Z}_d, \mathcal{R}\mathcal{Y}_b)$$

Using (3),(5), and (27), we get

$$\begin{aligned}\breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Z}_d), \mathcal{S}\mathcal{Y}_b) &= -\breve{g}_1((\bar{\nabla}_{\mathcal{X}_a}\mathcal{P})\mathcal{Z}_d, \mathcal{Y}_b) + \breve{g}_1(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P}\mathcal{Z}_d, \mathcal{Y}_b) \\ &- (\mathcal{Z}_d lnk)\breve{g}_1(\mathcal{X}_a, \mathcal{R}\mathcal{Y}_b)\end{aligned}$$

Using (10) and property (m_3) of $\mathcal{G}$, we obtain

$$\begin{aligned} \breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Z}_d), \mathcal{S}\mathcal{Y}_b) &= -\breve{g}_1(\mathcal{G}_{\mathcal{X}_a}\mathcal{Z}_d, \mathcal{Y}_b) - \breve{g}_1(\mathcal{A}_{\mathcal{P}\mathcal{Z}_d}\mathcal{X}_a, \mathcal{Y}_b) \\ &- (\mathcal{Z}_d lnk)\breve{g}_1(\mathcal{X}_a, \mathcal{R}\mathcal{Y}_b) \end{aligned}$$

$$\breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Z}_d), \mathcal{S}\mathcal{Y}_b) = -\breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Y}_b), \mathcal{P}\mathcal{Z}_d) - \breve{g}_1(\mathcal{G}_{\mathcal{X}_a}\mathcal{Z}_d, \mathcal{Y}_b) - (\mathcal{Z}_d lnk)\breve{g}_1(\mathcal{X}_a, \mathcal{R}\mathcal{Y}_b) \tag{30}$$

Interchanging $\mathcal{X}_a$ and $\mathcal{Y}_b$ in (30), we obtain

$$\breve{g}_1(h_1(\mathcal{Y}_b, \mathcal{Z}_d), \mathcal{S}\mathcal{X}_a) = -\breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Y}_b), \mathcal{P}\mathcal{Z}_d) - \breve{g}_1(\mathcal{G}_{\mathcal{Y}_b}\mathcal{Z}_d, \mathcal{X}_a) + (\mathcal{Z}_d lnk)\breve{g}_1(\mathcal{X}_a, \mathcal{R}\mathcal{Y}_b) \tag{31}$$

Thus, from (30) and (31), we obtain (29)

Lemma 4.3 *Let $\mathcal{N}_x = \mathcal{N}_a^\theta \times_k \mathcal{N}_b^\perp$ be a warped product pointwise hemislant type1-2 submanifold whose ambient spaces are nearly para-Kaehler manifold $\bar{\mathcal{N}}_x$. Then*

1) *For type-1-2;*

$$\begin{aligned} 2\breve{g}_1(h_1(\mathcal{Z}_d, \mathcal{W}_c), S\mathcal{X}_a) &= -\breve{g}_1(h_1(\mathcal{X}_a, \mathcal{Z}_d), \mathcal{P}\mathcal{W}_c) - \breve{g}_1(h_1(\mathcal{X}_a, \mathcal{W}_c), \mathcal{P}\mathcal{Z}_d) \\ &+ 2(R\mathcal{X}_a lnk)\breve{g}_1(\mathcal{Z}_d, \mathcal{W}_c) \end{aligned} \tag{32}$$

2) a) *For type-1;*

$$\begin{aligned} 2\breve{g}_1(h_1(\mathcal{Z}_d, \mathcal{W}_c), SR\mathcal{X}_a) &= -\breve{g}_1(h_1(R\mathcal{X}_a, \mathcal{Z}_d), \mathcal{P}\mathcal{W}_c) - \breve{g}_1(h_1(R\mathcal{X}_a, \mathcal{W}_c)\mathcal{P}\mathcal{Z}_d) \\ &- 2cosh^2\theta(\mathcal{X}_a lnk)\breve{g}_1(\mathcal{Z}_d, \mathcal{W}_c) \end{aligned} \tag{33}$$

b) *For type-2 ;*

$$\begin{aligned} \breve{g}_1(h_1(\mathcal{Z}_d, \mathcal{W}_c), SR\mathcal{X}_a) &= -\breve{g}_1(h_1(R\mathcal{X}_a, \mathcal{Z}_d), \mathcal{P}\mathcal{W}_c) - \breve{g}_1(h_1(R\mathcal{X}_a, \mathcal{W}_c), \mathcal{P}\mathcal{Z}_d) \\ &- 2cos^2\theta(\mathcal{X}_a lnk)\breve{g}_1(\mathcal{Z}_d, \mathcal{W}_c) \end{aligned} \tag{34}$$

for any $\mathcal{Z}_d, \mathcal{W}_c \in T\mathcal{N}_b^\perp$ and $\mathcal{X}_a \in T\mathcal{N}_a^\theta$.

Proof. Using (8) and (5), we get

$$\begin{aligned} \breve{g}_1(h_1(\mathcal{Z}_d, \mathcal{W}_c), S\mathcal{X}_a) &= \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{W}_c, \mathcal{P}\mathcal{X}_a) - \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{W}_c, \mathcal{R}\mathcal{X}_a) \\ &= -\breve{g}_1(\mathcal{P}\bar{\nabla}_{\mathcal{Z}_d}\mathcal{W}_c, \mathcal{X}_a) - \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{W}_c, \mathcal{R}\mathcal{X}_a) \end{aligned}$$

By using (2),(3) (7), (10), (27) and $\mathcal{W}_c$ and $\mathcal{R}\mathcal{X}_a$ are orthogonality, we get

$$\begin{aligned} \breve{g}_1(h_1(\mathcal{Z}_d, \mathcal{W}_c), S\mathcal{X}_a) &= -\breve{g}_1((\bar{\nabla}_{\mathcal{Z}_d}\mathcal{P})\mathcal{W}_c, \mathcal{X}_a) + \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{P}\mathcal{W}_c, \mathcal{X}_a) \\ &+ \breve{g}_1(\mathcal{W}_c, \bar{\nabla}_{\mathcal{Z}_d}R\mathcal{X}_a) \\ &= -\breve{g}_1(\mathcal{G}_{\mathcal{Z}_d}\mathcal{W}_c, \mathcal{X}_a) - \breve{g}_1(A_{\mathcal{P}\mathcal{W}_c}\mathcal{Z}_d, \mathcal{X}_a) \\ &+ (R\mathcal{X}_a lnk)\breve{g}_1(\mathcal{Z}_d, \mathcal{W}_c) \\ &= -\breve{g}_1(\mathcal{G}_{\mathcal{Z}_d}\mathcal{W}_c, \mathcal{X}_a) - \breve{g}_1((h_1(\mathcal{X}_a, \mathcal{Z}_d), \mathcal{P}\mathcal{W}_c) \\ &+ (R\mathcal{X}_a lnk)\breve{g}_1(\mathcal{Z}_d, \mathcal{W}_c) \end{aligned} \tag{35}$$

Interchanging $\mathcal{Z}_d$ and $\mathcal{W}_c$ in 29, we obtain

$$\begin{aligned} \breve{g}_1(h_1(\mathcal{Z}_d,\mathcal{W}_c),S\mathcal{X}_a) &= -\breve{g}_1(\mathcal{G}_{\mathcal{W}_c},\mathcal{Z}_d\mathcal{X}_a)-\breve{g}_1((h_1(\mathcal{X}_a,\mathcal{W}_c),\mathcal{P}\mathcal{Z}_d) \\ &+ (R\mathcal{X}_a lnk)\breve{g}_1(\mathcal{Z}_d,\mathcal{W}_c) \end{aligned} \tag{36}$$

From (35) and (36), we derive 1. If we interchange $\mathcal{X}_a$ by $R\mathcal{X}_a$ in (1), we obtain (2) (for type-1). In a similar way, the proof for type-2 is obtained.

Theorem 4.4. *Let $\mathcal{N}_x = \mathcal{N}_a^\theta \times_k \mathcal{N}_b^\perp$ be a warped product pointwise hemislant type1-2 submanifold whose ambient spaces are nearly para-Kaehler manifolds $\bar{\mathcal{N}}_x$. So that $\mathcal{G}_{\mathcal{Z}_d}\mathcal{W}_c \in \Gamma(D_t^\perp)$, for any $\mathcal{Z}_d,\mathcal{W}_c \in \Gamma(D_t^\perp)$ and $\mathcal{U}_{\mathcal{L}_1}\mathcal{V}_f \in \Gamma(\mu)$, for any $\mathcal{L}_1,\mathcal{V}_f \in \Gamma(\mathcal{T}\mathcal{N}_x)$ where $\mathcal{D}_t^\perp$ and μ are totally real distribution and invariant normal subbundle of $\mathcal{N}_x$, respectively. Then $\mathcal{N}_x$ is locally a mixed geodesic warped product pointwise submanifold of the form $\mathcal{N}_a^\theta \times_k \mathcal{N}_b^\perp$ if and only if*
**For type-1,*

$$A_{\mathcal{P}\mathcal{Z}_d}\mathcal{X}_a = 0 \quad and \quad A_{SR\mathcal{X}_a}\mathcal{Z}_a = cosh^2\theta\mathcal{X}_a(\varphi)\mathcal{Z}_d \tag{37}$$

**For type-2,*

$$A_{\mathcal{P}\mathcal{Z}_d}\mathcal{X}_a = 0, \quad and \quad A_{SR\mathcal{X}_a}\mathcal{Z}_d = -cos^2\theta\mathcal{X}_a(\varphi)\mathcal{Z}_d \tag{38}$$

for any $\mathcal{X}_a \in D_n^\alpha$ and $\mathcal{Z}_d \in D_t^\perp$, that φ is a function on $\mathcal{N}_x$ so $\mathcal{W}'_c(\varphi)=0$, for any $\mathcal{W}'_c \in \Gamma(D_t^\perp)$.

Proof. Let $\mathcal{N}_x = \mathcal{N}_a^\theta \times_k \mathcal{N}_b^\perp$ be a warped product pointwise hemislant type1-2 submanifold whose ambient spaces are nearly para-Kaehler manifolds $\bar{\mathcal{N}}_x$. So that $\mathcal{N}_a^\theta$ and $\mathcal{N}_b^\perp$ proper pointwise slant and totally real submanifolds of $\bar{\mathcal{N}}_x$. In that case, by using (24),(25) and (26), we derive (37) and (38).

Conversely, if $\mathcal{N}_x$ is a proper hemislant submanifold whose ambient spaces are nearly para-Kaehler manifolds $\bar{\mathcal{N}}_x$. So that $\mathcal{G}_{\mathcal{Z}_d}\mathcal{W}_c \in \Gamma(D_t^\perp)$, for any $\mathcal{Z}_d,\mathcal{W}_c \in \Gamma(D_t^\perp)$ and $\mathcal{U}_{\mathcal{L}_1}\mathcal{V}_f \in \Gamma(\mu)$, for any $\mathcal{L}_1,\mathcal{V}_f \in \Gamma(\mathcal{T}\mathcal{N}_x)$. Then, using (27) and the relation (37), (38), we obtain $\breve{g}_1(\nabla_{\mathcal{X}_a}\mathcal{Y}_b,\mathcal{Z}_d)=0$, implying that the leaves of D_n^α are totally geodesic in $\mathcal{N}_x$.
For any $\mathcal{Z}_d,\mathcal{W}_c \in \Gamma(D_t^\perp)$ and for any $\mathcal{X}_a \in D_n^\alpha$, we obtain

$$\begin{aligned} \breve{g}_1([\mathcal{Z}_d,\mathcal{W}_c],\mathcal{X}_a) &= \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{W}_c,\mathcal{X}_a)-\breve{g}_1(\bar{\nabla}_{\mathcal{W}_c}\mathcal{Z}_d,\mathcal{X}_a) \\ &= -\breve{g}_1(\mathcal{P}\bar{\nabla}_{\mathcal{Z}_d}\mathcal{W}_c,\mathcal{P}\mathcal{X}_a)+\breve{g}_1(\mathcal{P}\bar{\nabla}_{\mathcal{W}_c}\mathcal{Z}_d,\mathcal{P}\mathcal{X}_a) \\ &= -\breve{g}_1(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{P}\mathcal{W}_c,\mathcal{P}\mathcal{X}_a)+\breve{g}_1((\bar{\nabla}_{\mathcal{Z}_d}\mathcal{P})\mathcal{W}_c,\mathcal{P}\mathcal{X}_a) \\ &+ \breve{g}_1(\bar{\nabla}_{\mathcal{W}_c}\mathcal{P}\mathcal{Z}_d,\mathcal{P}\mathcal{X}_a)-\breve{g}_1((\bar{\nabla}_{\mathcal{W}_c}\mathcal{P})\mathcal{Z}_d,\mathcal{P}\mathcal{X}_a) \\ &= +\breve{g}_1(\mathcal{P}\mathcal{W}_c,\bar{\nabla}_{\mathcal{Z}_d}\mathcal{P}\mathcal{X}_a)+\breve{g}_1(\mathcal{G}_{\mathcal{Z}_d}\mathcal{W}_c,\mathcal{R}\mathcal{X}_a)+\breve{g}_1(\mathcal{U}_{\mathcal{Z}_d}\mathcal{W}_c,\mathcal{S}\mathcal{X}_a) \\ &- \breve{g}_1(\mathcal{P}\mathcal{Z}_d,\bar{\nabla}_{\mathcal{W}_c}\mathcal{P}\mathcal{X}_a)-\breve{g}_1(\mathcal{G}_{\mathcal{W}_c}\mathcal{Z}_d,\mathcal{R}\mathcal{X}_a)-\breve{g}_1(\mathcal{U}_{\mathcal{W}_c}\mathcal{Z}_d,\mathcal{S}\mathcal{X}_a) \end{aligned}$$

Since $\mathcal{G}_{\mathcal{Z}_d}\mathcal{W}_c \in \Gamma(D_t^{\perp})$, for any $\mathcal{Z}_d, \mathcal{W}_c \in \Gamma(D_t^{\perp})$ and $\mathcal{U}_{\mathcal{L}_1}\mathcal{V}_f \in \Gamma(\mu)$, for any $\mathcal{L}_1, \mathcal{V}_f \in \Gamma(\mathcal{T}\mathcal{N}_x)$, Then

$$\begin{aligned}
\breve{g}_1([\mathcal{Z}_d,\mathcal{W}_c],\mathcal{X}_a) &= +\breve{g}_1(\mathcal{P}\mathcal{W}_c,\bar{\nabla}_{\mathcal{Z}_d}\mathcal{R}\mathcal{X}_a)+\breve{g}_1(\mathcal{P}\mathcal{W}_c,\bar{\nabla}_{\mathcal{Z}_d}\mathcal{S}\mathcal{X}_a)\\
&- \breve{g}_1(\mathcal{P}\mathcal{Z}_d,\bar{\nabla}_{\mathcal{W}_c}\mathcal{R}\mathcal{X}_a)-\breve{g}_1(\mathcal{P}\mathcal{Z}_d,\bar{\nabla}_{\mathcal{W}_c}\mathcal{S}\mathcal{X}_a)\\
&= -\breve{g}_1(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{P}\mathcal{S}\mathcal{X}_a,\mathcal{W}_c)+\breve{g}_1((\bar{\nabla}_{\mathcal{Z}_d}\mathcal{P})\mathcal{S}\mathcal{X}_a,\mathcal{W}_c)\\
&+ \breve{g}_1(h_1(\mathcal{Z}_d,\mathcal{R}\mathcal{X}_a),\mathcal{P}\mathcal{W}_c)+\breve{g}_1(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{P}\mathcal{S}\mathcal{X}_a,\mathcal{W}_c)\\
&- \breve{g}_1((\bar{\nabla}_{\mathcal{Z}_d}\mathcal{P})\mathcal{S}\mathcal{X}_a,\mathcal{W}_c)-\breve{g}_1(h_1(\mathcal{W}_c,\mathcal{R}\mathcal{X}_a),\mathcal{P}\mathcal{Z}_d)
\end{aligned}$$

Using (8),(9), and (10), we derive

$$\begin{aligned}
\breve{g}_1([\mathcal{Z}_d,\mathcal{W}_c],\mathcal{X}_a) &= -\breve{g}_1(\bar{\nabla}_{\mathcal{Z}_d}r\mathcal{S}\mathcal{X}_a,\mathcal{W}_c)-\breve{g}_1(\bar{\nabla}_{\mathcal{Z}_d}s\mathcal{S}\mathcal{X}_a,\mathcal{W}_c)\\
&+ \breve{g}_1(\mathcal{G}_{\mathcal{Z}_d}\mathcal{S}\mathcal{X}_a,\mathcal{W}_c)+\breve{g}_1(A_{\mathcal{P}\mathcal{W}_c}\mathcal{R}\mathcal{X}_a),\mathcal{Z}_d\\
&- \breve{g}_1(\bar{\nabla}_{\mathcal{W}_c}r\mathcal{S}\mathcal{X}_a,\mathcal{Z}_d)-\breve{g}_1(\bar{\nabla}_{\mathcal{W}_c}s\mathcal{S}\mathcal{X}_a,\mathcal{Z}_d)\\
&+ \breve{g}_1(\mathcal{G}_{\mathcal{W}_c}\mathcal{S}\mathcal{X}_a,\mathcal{Z}_d)+\breve{g}_1(A_{\mathcal{P}\mathcal{Z}_d}\mathcal{R}\mathcal{X}_a,\mathcal{W}_c)
\end{aligned}$$

Using (18), property $m_3(b)$, and (37), we derive

$$\begin{aligned}
\breve{g}_1([\mathcal{Z}_d,\mathcal{W}_c],\mathcal{X}_a) &= sinh^2\theta\breve{g}_1(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{X}_a,\mathcal{W}_c)+2sinh\theta cosh\theta\mathcal{Z}_d(\theta)\breve{g}_1(\mathcal{X}_a,\mathcal{W}_c)\\
&+ \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{S}\mathcal{R}\mathcal{X}_a,\mathcal{W}_c)-\breve{g}_1(\mathcal{U}_{\mathcal{Z}_d}\mathcal{W}_c,\mathcal{S}\mathcal{X}_a)\\
&- sinh^2\theta\breve{g}_1(\bar{\nabla}_{\mathcal{W}_c}\mathcal{X}_a,\mathcal{Z}_d)-2sinh\theta cosh\theta\mathcal{W}_c(\theta)\breve{g}_1(\mathcal{X}_a,\mathcal{Z}_d)\\
&- \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{S}\mathcal{R}\mathcal{X}_a,\mathcal{W}_c)+\breve{g}_1(\mathcal{U}_{\mathcal{W}_c}\mathcal{Z}_d,\mathcal{S}\mathcal{X}_a)
\end{aligned}$$

Using $\mathcal{G}_{\mathcal{Z}_d}\mathcal{W}_c \in \Gamma(D_t^{\perp})$, for any $\mathcal{Z}_d, \mathcal{W}_c \in \Gamma(D_t^{\perp})$, $\mathcal{U}_{\mathcal{L}_1}\mathcal{V}_f \in \Gamma(\mu)$, for any $\mathcal{L}_1, \mathcal{V}_f \in \Gamma(\mathcal{T}\mathcal{N}_x)$ and then by (5), (6), we derive

$$\begin{aligned}
\breve{g}_1([\mathcal{Z}_d,\mathcal{W}_c],\mathcal{X}_a) &= -sinh^2\theta\breve{g}_1([\mathcal{Z}_d,\mathcal{W}_c],\mathcal{X}_a)\\
&- \breve{g}_1(A_{\mathcal{S}\mathcal{R}\mathcal{X}_a}\mathcal{Z}_d,\mathcal{W}_c)+\breve{g}_1(A_{\mathcal{S}\mathcal{R}\mathcal{X}_a}\mathcal{W}_c,\mathcal{Z}_d)
\end{aligned}$$

Since $A_{\mathcal{S}\mathcal{R}\mathcal{X}_a}$ is symmetric and $\theta \neq \frac{\pi}{2}$, we obtain $\mathcal{D}_t^{\perp}$ is integrable. If we imagine that $\mathcal{N}_b^{\perp}$ is a leaf of $\mathcal{D}_t^{\perp}$ in $\mathcal{N}_x$, $h^{\perp}$ be a second fundamental form of $\mathcal{N}_b^{\perp}$ in $\mathcal{N}_x$. Then $\breve{g}_1(h^{\perp}(\mathcal{Z}_d,\mathcal{W}_c),\mathcal{X}_a)=\breve{g}_1(\nabla_{\mathcal{Z}_d}\mathcal{W}_c,\mathcal{X}_a)=\breve{g}_1(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{W}_c,\mathcal{X}_a)$. Using (1) and (10), we get

$$\begin{aligned}
\breve{g}_1(h^{\perp}(\mathcal{Z}_d,\mathcal{W}_c),\mathcal{X}_a) &= \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{W}_c,\mathcal{X}_a)\\
&= -\breve{g}_1((\bar{\nabla}_{\mathcal{Z}_d}\mathcal{P}\mathcal{W}_c,\mathcal{P}\mathcal{X}_a)\\
&= -\breve{g}_1(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{P}\mathcal{W}_c,\mathcal{P}\mathcal{X}_a)+\breve{g}_1((\bar{\nabla}_{\mathcal{Z}_d}\mathcal{P})\mathcal{W}_c,\mathcal{P}\mathcal{X}_a)\\
&= +\breve{g}_1(\mathcal{P}\mathcal{W}_c,\bar{\nabla}_{\mathcal{Z}_d}\mathcal{P}\mathcal{X}_a)+\breve{g}_1(\mathcal{G}_{\mathcal{Z}_d}\mathcal{W}_c,\mathcal{R}\mathcal{X}_a)+\breve{g}_1(\mathcal{U}_{\mathcal{Z}_d}\mathcal{W}_c,\mathcal{S}\mathcal{X}_a)
\end{aligned}$$

From $\mathcal{G}_{\mathcal{Z}_d}\mathcal{W}_c \in \Gamma(D_t^{\perp})$, for any $\mathcal{Z}_d, \mathcal{W}_c \in \Gamma(D_t^{\perp})$ and $\mathcal{U}_{\mathcal{L}_1}\mathcal{V}_f \in \Gamma(\mu)$, for any $\mathcal{L}_1, \mathcal{V}_f \in \Gamma(\mathcal{T}\mathcal{N}_x)$ and using (5),(8),(9), and (10), we obtain

$$\begin{aligned}
\breve{g}_1(h^{\perp}(\mathcal{Z}_d,\mathcal{W}_c),\mathcal{X}_a) &= +\breve{g}_1(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{R}\mathcal{X}_a,\mathcal{P}\mathcal{W}_c)-\breve{g}_1(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{P}\mathcal{S}\mathcal{X}_a,\mathcal{W}_c)\\
&+ \breve{g}_1((\bar{\nabla}_{\mathcal{Z}_d}\mathcal{P})\mathcal{S}\mathcal{X}_a,\mathcal{W}_c)\\
&= \breve{g}_1(h_1(\mathcal{Z}_d,\mathcal{R}\mathcal{X}_a),\mathcal{P}\mathcal{W}_c)-\breve{g}_1(\bar{\nabla}_{\mathcal{Z}_d}r\mathcal{S}\mathcal{X}_a,\mathcal{W}_c)\\
&- \breve{g}_1(\bar{\nabla}_{\mathcal{Z}_d}s\mathcal{S}\mathcal{X}_a,\mathcal{W}_c)+\breve{g}_1(\mathcal{G}_{\mathcal{Z}_d}\mathcal{S}\mathcal{X}_a,\mathcal{W}_c)
\end{aligned}$$

Using property $(m_3)(b)$, (5),(6),(4), and (18) , we derive

$$\begin{aligned}
\breve{g}_1(h^{\perp}(\mathcal{Z}_d,\mathcal{W}_c),\mathcal{X}_a) &= -\breve{g}_1(A_{\mathcal{P}\mathcal{W}_c}R\mathcal{X}_a,\mathcal{Z}_d)-sinh^2\theta\breve{g}_1(\bar{\nabla}_{\mathcal{Z}_d}\mathcal{W}_c,\mathcal{X}_a)\\
&+ \breve{g}_1(A_{\mathcal{S}\mathcal{R}\mathcal{X}_a}\mathcal{Z}_d,\mathcal{W}_c)-\breve{g}_1(\mathcal{U}_{\mathcal{Z}_d}\mathcal{W}_c,\mathcal{S}\mathcal{X}_a)
\end{aligned}$$

Thus, by the hypothesis of the theorem, we obtain

$$cosh^2\theta\breve{g}_1(h^{\perp}(\mathcal{Z}_d,\mathcal{W}_c),\mathcal{X}_a) = cosh^2\theta\mathcal{X}_a(\varphi)\breve{g}_1(\mathcal{Z}_d,\mathcal{W}_c)$$

From the description of gradient, we get

$$\breve{g}_1(h^{\perp}(\mathcal{Z}_d,\mathcal{W}_c),\mathcal{X}_a) = \breve{g}_1(\mathcal{Z}_d,\mathcal{W}_c)\breve{g}_1(grad\varphi,\mathcal{X}_a)$$

so that $h^{\perp}(\mathcal{Z}_d,\mathcal{W}_c)=\breve{g}_1(\mathcal{Z}_d,\mathcal{W}_c)\breve{g}_1 grad\varphi$, for vectors $\mathcal{Z}_d,\mathcal{W}_c\in D_t^{\perp}$. Mean curvature is $H=grad\varphi$ and $\mathcal{N}_b^{\perp}$ is totally umbilical in $\mathcal{N}_x$,
where $grad\varphi$ is the gradient of φ $(\nabla\varphi)$ and φ is a nonconstant function on $\mathcal{N}_x$.
Now, we explain $grad\varphi$ is parallel corresponding to the normal connection $\mathcal{D}_t^{\perp}$ of $\mathcal{N}_b^{\perp}$ in $\mathcal{N}_x$. For $\mathcal{X}_a\in\mathcal{D}_n^{\alpha}$ and $\mathcal{W}_c\in D_t^{\perp}$, we get

$$\begin{aligned}
\breve{g}_1(D_{\mathcal{W}_c}grad\varphi,\mathcal{X}_a) &= \breve{g}_1(\nabla_{\mathcal{W}_c}grad\varphi,\mathcal{X}_a)\\
&= \mathcal{W}_c\breve{g}_1(grad\varphi,\mathcal{X}_a)-\breve{g}_1(grad\varphi,\nabla_{\mathcal{W}_c}\mathcal{X}_a)\\
&= \mathcal{W}_c(\mathcal{X}_a(\varphi))-\breve{g}_1(grad\varphi,[\mathcal{W}_c,\mathcal{X}_a])-\breve{g}_1(grad\varphi,\nabla_{\mathcal{X}_a}\mathcal{W}_c)\\
&= \mathcal{X}_a(\mathcal{W}_c\varphi)+\breve{g}_1(\nabla_{\mathcal{X}_a}grad\varphi,\mathcal{W}_c)\\
&= 0
\end{aligned}$$

Thus, $\mathcal{W}_c\varphi=0$, for every $\mathcal{W}_c\in\mathcal{D}_t^{\perp}$ also $\nabla_{\mathcal{X}_a}grad\varphi\in\mathcal{D}_n^{\alpha}$; therefore, $\mathcal{D}_n^{\alpha}$ is totally geodesic. We understand that mean curvature of $\mathcal{N}_b^{\perp}$ is parallel. Therefore, the leaves of $\mathcal{D}_t^{\perp}$ are totally umbilical with parallel mean curvature $H=grad\varphi$. Thus, $\mathcal{N}_b^{\perp}$ is called the extrinsic sphere in $\mathcal{N}_x$. By considering Hiepko [7], we attain that $\mathcal{N}_x$ is a warped product pointwise submanifold and (for type-*1*).
We obtain the proof (for type-1). Similarly, the proof for type-2 is obtained.

5. An optimal inequality

We achieve a connection for the squared norm of the second fundamental form. For later utilize, we give an orthonormal frame.

Let $\mathcal{N}_x = \mathcal{N}_a^\theta \times_k \mathcal{N}_b^\perp$ be an s-dimensional warped product pointwise hemislant submanifold whose ambient spaces are $(2m)$-dimensional nearly para-Kaehler manifold $\bar{\mathcal{N}_x}$ with $\mathcal{N}_b^\perp$ of dimension d_1 and basis $\mathcal{N}_a^\theta$ of dimension $d_2 = 2p$. We take tangent spaces of $\mathcal{N}_b^\perp$ and $\mathcal{N}_a^\theta$ by $\mathcal{D}_t^\perp$ and $\mathcal{D}_n^\alpha$. We create orthonormal frames according to type-1 and type-2. Firstly, for type-1, we create the orthonormal frames of $\mathcal{D}_n^\alpha$ and $\mathcal{D}_t^\perp$, respectively;

$\{\mathtt{E}_1, .., \mathtt{E}_p, \mathtt{E}_{p+1} = sech\theta\mathcal{T}\mathtt{E}_1, ..., \mathtt{E}_{2p} = sech\theta\mathcal{T}\mathtt{E}_p\}$ and

$\{\mathtt{E}_{2p+1} = \mathtt{E}_1^*, ..., \mathtt{E}_s = \mathtt{E}_{2p+d_1} = \mathtt{E}_{d_1}^*\}$ that θ is nonconstant.

At the moment, we will give orthonormal frames of the normal subbundles of SD_n^α ,$\mathcal{P}D_t^\perp$, μ. These frames are

$\{\mathtt{E}_{s+1} = \bar{\mathtt{E}}_{d_1+1} = csch\theta S\mathtt{E}_1^*, \mathtt{E}_{s+d_1+2} = \bar{\mathtt{E}}_1 = csch\theta S\mathtt{E}_1, ..., \mathtt{E}_{s+p} = \bar{\mathtt{E}}_p = csch\theta S\mathtt{E}_p,$

$\mathtt{E}_{s+p+1} = \bar{\mathtt{E}}_{p+1} = csch\theta sech\theta SR\mathtt{E}_1, ..., \mathtt{E}_{s+2p} = \bar{\mathtt{E}}_{2p} = csch\theta sech\theta SR\mathtt{E}_p\}$,

$\{\mathtt{E}_{s+2p+1} = \mathcal{P}\mathtt{E}_1^*, ..., \mathtt{E}_{2s} = \mathcal{P}\mathtt{E}_{d_1}^*\}$ and

$\{\mathtt{E}_{2s+1}, ..., \mathtt{E}_{2m}\}$

where θ is the slant function and SD_n^α ,$\mathcal{P}D_t^\perp$ and μ, respectively are $2p$, d_1 and $2m - 2s$

Let us assume that

* on $\mathcal{D}_t^\perp$: orthonormal basis $\{\mathtt{E}_v\}_{v=1,...,d_1}$, where $d_1 = dim(\mathcal{D}_t^\perp)$; also, supposed that $\breve{g}_1(\mathtt{E}_v, \mathtt{E}_v) = 1$.
* on $\mathcal{D}_n^\alpha$: orthonormal basis $\{\mathtt{E}_w^*\}_{w=1,...,2p}$, where $2p = dim(\mathcal{D}_n^\alpha)$ also $\breve{g}_1(\mathtt{E}_w^*, \mathtt{E}_w^*) = \mp 1$.
* on $\mathcal{P}\mathcal{D}_t^\perp$: orthonormal basis $\{\mathtt{E}_v\}_{v=1,...,d_1}$, where $d_1 = dim\mathcal{P}(\mathcal{D}_t^\perp)$ also $\breve{g}_1(\mathcal{P}\mathtt{E}_v, \mathcal{P}\mathtt{E}_v) = -1$.
* on $\mathcal{S}\mathcal{D}_n^\alpha$: orthonormal basis $\{\mathtt{E}_w^*\}_{w=1,...,2p}$, where $2p = dim\mathcal{S}(\mathcal{D}_n^\alpha)$ also $\breve{g}_1(\mathtt{E}_w^*, \mathtt{E}_w^*) = \mp 1$.

Theorem 5.1. *Let $\mathcal{N}_x = \mathcal{N}_a^\theta \times_k \mathcal{N}_b^\perp$ be an s-dimensional mixed geodesic warped product pointwise hemislant type-1 submanifold whose ambient spaces are $(2m)$- dimensional nearly para-Kaehler manifolds $\bar{\mathcal{N}_x}$. In this place, $\mathcal{N}_a^\theta$ is a proper pointwise slant submanifold and $\mathcal{N}_b^\perp$ is an antiinvariant submanifold of dimension d_1 of $\bar{\mathcal{N}_x}$. For $\mathcal{N}_b^\perp$ is spacelike, we get*

1) The squared norm of the second fundamental form of $\mathcal{N}_x$ supplies

$$||h_1||^2 \leq d_1 coth^2\theta ||grad(lnk)||^2 \tag{39}$$

where $grad(lnk)$ is the gradient of lnk $(\nabla(lnk))$ and k is a nonconstant function.

2) If the equality sign of (39) holds the same way, $\mathcal{N}_a^\theta$ is totally geodesic and $\mathcal{N}_b^\perp$ is totally umbilical in $\bar{\mathcal{N}_x}$.

Proof. From description $||h_1||^2 = ||h_1(\mathcal{D}_m, \mathcal{D}_m)||^2 + 2||h_1(\mathcal{D}_m, \mathcal{D}_t^\perp)||^2 + ||h_1(\mathcal{D}_t^\perp, \mathcal{D}_t^\perp)||^2$, it is seen that $\mathcal{D}_m = \mathcal{D}_n^\alpha$. Since $\mathcal{N}_x$ is mixed geodesic, the middle term of the right-hand side should vanish. In that case, we get

$$||h_1||^2 = \sum_{r=s+1}^{2m} \sum_{v,w=1}^{2p} \breve{g}_1(h_1(\mathtt{E}_v, \mathtt{E}_w), \mathtt{E}_r)^2 + \sum_{r=s+1}^{2m} \sum_{v,w=1}^{d_1} \breve{g}_1(h_1(\mathtt{E}_v^*, \mathtt{E}_w^*), \mathtt{E}_r)^2$$

This equation can be separated for the $\mathcal{P}D_t^\perp$, SD_n^α and μ components as follows:

$$\begin{aligned} ||h_1||^2 &= \sum_{r=1}^{2p}\sum_{v,w=1}^{2p}\breve{g}_1(h_1(\mathtt{E}_v,\mathtt{E}_w),\bar{\mathtt{E}}_r)^2+\sum_{r=1}^{d_1}\sum_{v,w=1}^{2p}\breve{g}_1(h_1(\mathtt{E}_v,\mathtt{E}_w),\mathcal{P}\mathtt{E}_r^*)^2 \\ &+ \sum_{r=2s+1}^{2m}\sum_{v,w=1}^{2p}\breve{g}_1(h_1(\mathtt{E}_v,\mathtt{E}_w),\mathtt{E}_r)^2+\sum_{r=1}^{2p}\sum_{v,w=1}^{d_1}\breve{g}_1(h_1(\mathtt{E}_v^*,\mathtt{E}_w^*),\bar{\mathtt{E}}_r)^2 \\ &+ \sum_{r=1}^{d_1}\sum_{v,w=1}^{d_1}\breve{g}_1(h_1(\mathtt{E}_v^*,\mathtt{E}_w^*),\mathcal{P}\mathtt{E}_r^*)^2 \\ &+ \sum_{r=2s+1}^{2m}\sum_{v,w=1}^{d_1}\breve{g}_1(h_1(\mathtt{E}_v^*,\mathtt{E}_w^*),\mathtt{E}_r)^2 \end{aligned} \tag{40}$$

Utilizing (29), the second term of the right-hand side in the last equation is zero for a mixed geodesic. This is because there is no relationship in terms of the warping function for the first, third, fifth, and sixth terms in the right hand side of the last equation. Therefore, these terms vanish, leaving only the fourth term to be evaluated.

$$\begin{aligned} ||h_1||^2 &\leq \sum_{r=1}^{p}\sum_{v,w=1}^{d_1}\breve{g}_1(h_1(\mathtt{E}_v^*,\mathtt{E}_w^*),csch\theta S\mathtt{E}_r)^2 \\ &+ \sum_{r=1}^{p}\sum_{v,w=1}^{d_1}\breve{g}_1(h_1(\mathtt{E}_v^*,\mathtt{E}_w^*),csch\theta sech\theta SR\mathtt{E}_r)^2 \end{aligned}$$

Using (28),(29), and (32), we get

$$\begin{aligned}
||h_1||^2 &\leq csch^2\theta \sum_{r=1}^{p}\sum_{v,w=1}^{d_1}(R\mathtt{E}_w lnk)^2\breve{g}_1(\mathtt{E}_v^*,\mathtt{E}_w^*)^2 \\
&+ coth^2\theta \sum_{r=1}^{p}\sum_{v,w=1}^{d_1}(\mathtt{E}_r lnk)^2\breve{g}_1(\mathtt{E}_v^*,\mathtt{E}_w^*)^2 \\
&= d_1 csch^2\theta \sum_{r=1}^{2p}(R\mathtt{E}_r lnk)^2 - d_1 csch^2\theta \sum_{r=p+1}^{2p}(R\mathtt{E}_r lnk)^2 \\
&+ d_1 coth^2\theta \sum_{r=1}^{p}(\mathtt{E}_r lnk)^2 \\
&= d_1 csch^2\theta ||Rgradlnk||^2 - d_1 csch^2\theta \sum_{r=1}^{p}\breve{g}_1(\mathtt{E}_{r+p}, Rgradlnk)^2 \\
&+ d_1 coth^2\theta \sum_{r=1}^{p}(\mathtt{E}_r lnk)^2 \\
&= d_1 coth^2\theta ||gradlnk||^2 - d_1 csch^2\theta sech^2\theta \sum_{r=1}^{p}\breve{g}_1(R\mathtt{E}_r, Rgradlnk)^2 \\
&+ d_1 coth^2\theta \sum_{r=1}^{p}(\mathtt{E}_r lnk)^2
\end{aligned}$$

By using (16), we obtain $||h_1||^2 \leq d_1 coth^2\theta ||gradlnk||^2$, which is inequality 1). If the equality holds in (39), then from the remaining first and third terms in (40), we derive

$$\breve{g}_1(h_1(\mathcal{D}_n^\alpha,\mathcal{D}_n^\alpha), S\mathcal{D}_n^\alpha) = 0 \Rightarrow h_1(\mathcal{D}_n^\alpha,\mathcal{D}_n^\alpha) \subset \mathcal{P}\mathcal{D}_t^\perp \oplus \mu \tag{41}$$

and

$$\breve{g}_1(h_1(\mathcal{D}_n^\alpha,\mathcal{D}_n^\alpha), \mu) = 0 \Rightarrow h_1(\mathcal{D}_n^\alpha,\mathcal{D}_n^\alpha) \subset \mathcal{P}\mathcal{D}_t^\perp \oplus S\mathcal{D}_n^\alpha \tag{42}$$

Then from (41) and (42), we obtain

$$h_1(\mathcal{D}_n^\alpha,\mathcal{D}_n^\alpha) \subset \mathcal{P}\mathcal{D}_t^\perp \tag{43}$$

However, using (32) for a mixed geodesic, we obtain

$$h_1(\mathcal{D}_n^\alpha,\mathcal{D}_n^\alpha) \perp \mathcal{P}\mathcal{D}_t^\perp \tag{44}$$

Then, from (43) and (44), we derive

$$h_1(\mathcal{D}_n^\alpha,\mathcal{D}_n^\alpha) = 0 \tag{45}$$

Because $\mathcal{N}_a^\theta$ is totally geodesic in $\mathcal{N}_x$ [3], (45) means that $\mathcal{N}_a^\theta$ is totally geodesic in $\bar{\mathcal{N}}_x$. From the remaining fifth and sixth terms, we express

$$\breve{g}_1(h_1(\mathcal{D}_t^{\perp}, \mathcal{D}_t^{\perp}), \mathcal{P}\mathcal{D}_t^{\perp}) = 0 \Rightarrow h_1(\mathcal{D}_t^{\perp}, \mathcal{D}_t^{\perp}) \subset S\mathcal{D}_n^{\alpha} \oplus \mu \tag{46}$$

$$\breve{g}_1(h_1(\mathcal{D}_t^{\perp}, \mathcal{D}_t^{\perp}), \mu) = 0 \Rightarrow h_1(\mathcal{D}_t^{\perp}, \mathcal{D}_t^{\perp}) \subset \mathcal{P}\mathcal{D}_t^{\perp} \oplus S\mathcal{D}_n^{\alpha} \tag{47}$$

Then from (46) and (47), we obtain

$$h_1(\mathcal{D}_t^{\perp}, \mathcal{D}_t^{\perp}) \subset S\mathcal{D}_n^{\alpha} \tag{48}$$

Also from (33) for a mixed geodesic, we obtain

$$\breve{g}_1(h_1(\mathcal{Z}_d, \mathcal{W}_c), \mathcal{S}\mathcal{R}\mathcal{X}_a) = -cosh^2\theta(\mathcal{X}_a lnk)\breve{g}_1(\mathcal{Z}_d, \mathcal{W}_c) \tag{49}$$

for any $\mathcal{X}_a \in T\mathcal{N}_a^{\theta}$ and $\mathcal{Z}_d, \mathcal{W}_c \in T\mathcal{N}_b^{\perp}$. Therefore, from (48) and (49) ,$\mathcal{N}_b^{\perp}$ is totally umbilical in $\mathcal{N}_x$ [3], we get that $\mathcal{N}_b^{\perp}$ is totally umbilical in $\bar{\mathcal{N}_x}$. The proof is obtained.

Remark 5.2 *If $\mathcal{N}_b^{\perp}$ manifold of Theorem 5.1 is totally umbilical and timelike, equation (39) should be modified by*

$$||h_1||^2 \geq d_1 coth^2\theta ||grad(lnk)||^2 \tag{50}$$

where $grad(lnk)$ is the gradient of lnk (∇ (lnk)) and k is a nonconstant function.
In a similar way, for proper pointwise slant submanifold $\mathcal{N}_a^{\theta}$ of type-2, we achieve

Theorem 5.3 *Let $\mathcal{N}_x = \mathcal{N}_a^{\theta} \times_k \mathcal{N}_b^{\perp}$ be an s-dimensional mixed geodesic warped product hemislant submanifold whose ambient spaces are* $(2\mathtt{m})$*- dimensional nearly para-Kaehler manifold $\bar{\mathcal{N}_x}$. Hence, $\mathcal{N}_b^{\perp}$ is spacelike and timelike, respectively. Then, (for type-2)*
The squared norm of the second fundamental form of $\mathcal{N}_x$ supplies:

$$||h_1||^2 \leq d_1 cot^2\theta ||grad(lnk)||^2 (respectively, ||h_1||^2 \geq d_1 cot^2\theta ||grad(lnk)||^2) \tag{51}$$

where $grad(lnk)$ is the gradient of lnk (∇ (lnk)) and k is a nonconstant function.

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